On the Clique Partition of Rectangle Graphs

Gilles Chabert¹ and Xavier Lorca¹
Ecole des Mines de Nantes LINA CNRS UMR 6241,
4, rue Alfred Kastler 44300 Nantes, France
gilles.chabert@emn.fr  xavier.lorca@emn.fr

Abstract. In this note, we show the NP-completeness of the clique partition problem for the class of rectangle graphs.

1 Introduction

This note deals with a particular class of intersection graphs, i.e., graphs in which vertices represent geometrical objects and edges non-empty intersections between objects. This class, called rectangle graphs represents intersecting rectangles in the plane.

Other classes of intersection graphs are frequently considered, such as disk graphs, unit disk graphs, interval graphs, circular arc graphs or square graphs (see [7] for a survey).

Rectangle graphs have turned to be important in the field of computational geometry (see, e.g., [1, 3]).

We prove in this note that the (vertex–)clique partition problem is NP-complete for rectangle graphs. This result has a meaningful consequence in the context of constraint programming as we will explain in conclusion.

1.1 Definitions and Problem Statement

A rectangle graph is a $n$-vertices $m$-edges graph $G_r = (V_r, E_r)$ that can be extracted from $n$ axis-aligned rectangles in the plane as follows:

– a vertex represents a rectangle
– an edge represents a non-empty intersection between two rectangles.

Figure 1 depicts the two possible views of a rectangle graph. Since any set of pairwise intersecting rectangles intersect all mutually (by Helly’s theorem), a $k$-clique in $G_r$ represents the intersection of $k$ rectangles in the geometrical view. For instance, consider the 3-clique $S1$ of $G_r$ of Figure 1a, it is represented by the intersection of three rectangles ($R4, R5, R6$) in Figure 1b, depicted by the black rectangle $S1$. For a rectangle graph $G_r$, the rectangle clique partition problem (RCP) is to find a vertex partition of size $k \leq n$ such that each element of the partition forms a complete graph. Formally:
Instance: A n-vertices m-edges rectangle graph $G_r = (V_r, E_r)$ given in the form of n axis-aligned rectangles in the plane and $k \leq n$.

Question: Can $V_r$ be partition into $k$ disjoint sets $V_1, \ldots, V_k$ such that $\forall i, 1 \leq i \leq k$ the subgraph induced by $V_i$ is a complete graph?

Consider another time Figure 1a, $G_r$ can be partitioned into 4 disjoint set of vertices, each forming a complete subgraph of $G_r$: $\{R_1, R_2\}$, $\{R_3\}$, $\{R_4, R_5, R_6\}$, $\{R_7\}$.

Lemma 1. Given a $k$–partition $V_1, \ldots, V_k$ of a rectangle graph $G_r = (V_r, E_r)$, such that $\forall i, 1 \leq i \leq k$, $V_i \subseteq V_r$, checking that each subgraph induced by $V_i$ is a complete graph can be done in polynomial time.

Proof. This can be trivially done by checking that for each $V_i$ in $V'$, the subgraph induced by $V_i$ has exactly $\frac{|V_i| \times (|V_i| - 1)}{2}$ edges. $\square$

In the following, we show that RCP is NP-complete by a reduction from the cubic planar vertex cover problem (CPVC) [10]. The difficulty lies in the fact that the class of rectangle graphs is only characterized by the geometrical interpretation. The transformation is inspired from that in [8] but contains important differences.

2 Preliminaries

First of all, let us give some definition related to some particular graphs classes (see Figure 2). An embedding of a graph $G$ on the plane is a representation of $G$ in which points are associated to vertices and arcs are associated to edges in such a way that:
– the endpoints of the arc associated to an edge $e$ are the points associated to the end vertices of $e$,
– no arcs include points associated with other vertices,
– two arcs never intersect at a point which is interior to either of the arcs.

A planar graph (Figure 2a) is a graph which admits an embedding on the plane (Figure 2b), and a cubic graph is a 3-regular graph, i.e., a graph in which every vertex has 3 incident edges. Finally, A rectilinear embedding is an embedding where every arc is a broken line formed by horizontal and vertical segments (Figure 2c).

Fig. 2: A (cubic) planar graph, one of its embedding on the plane, and one of its rectilinear embedding.

3 Transformation From Cubic Planar Vertex Cover

Consider the well-known NP-Complete vertex cover problem. Such a problem remains NP-Complete even for cubic planar graphs [10]. Formally, the Cubic Planar Vertex Cover problem (CPVC) is stated as follows:

**Instance:** A cubic planar graph $G_p = (V_p, E_p)$ and $k \leq |V_p|$.

**Question:** Is there a subset $V' \subseteq V_p$ with $|V'| \leq k$ such each edge of $E_p$ has at least one of its extremity in $V'$?

We prove now that the answer to CPVC is yes if and only if the answer to RCP is yes with $(G_r, k + 4 \times |E|)$ where $E$ is the number of edges in $G_p$.

We show a polytime transformation of an instance $(G_p, k)$ of CPVC into an instance $(G_r, k + 4 \times |E_p|)$ of RCP, where $E_p$ is the number of edges in $G_p$.

Note that the clique partition problem for cubic planar graphs is known to be NP-complete [5] but since the transformation that generates rectangle graphs below “breaks” the structure of cliques, this problem was actually not a good candidate for the reduction.
First, consider the polytime tranformation of a planar graph into a rectilinear embedding provided by [9]. Their transformation ensures that the number of bends per edges is bounded by 4. For instance, consider another time Figures 2a and 2c, the first one represent a planar graph while the second one its possible rectilinear embedding. Notice that edge \((p_2, p_3)\) requires one bend, while the edge \((p_1, p_3)\) requires two bends. Every cubic planar graphs (except the tetrahedron) is actually 1-rectilinear, i.e., admiss a rectilinear embedding where each arc has at most one bend [6]. But, to our knowledge, there is no polynomial algorithm for computing such an embedding.

Second, assume extra segments are added in the way that each edge is represented with a broken line formed by exactly five segments. Figure 3 depicts the new embedding provided by the transformation. Such a drawing is called a rectilinear segmented embedding of a graph in the following.

![Fig. 3: A rectilinear segmented embedding of the graph of Figure 2c where each edge is split into 5 segments (segments are delineated by arrows)](image)

Now, given a cubic planar graph \(G_p\), we detail how to build the geometrical view of a rectangle graph from the rectilinear segmented embedding graph associated with \(G_p\):

- Every segment of the rectilinear segmented embedding of the graph \(G_p\) is replaced by a rectangle such that if two segments do not intersect in the rectilinear segmented embedding, the corresponding rectangles do not intersect (should they be flat).
- The segments having a vertex at a common endpoint intersects all mutually in the neighborhood of this vertex (Figure 4a). Notice the case of a leaf node is irrelevant because \(G_p\) is cubic.
- Segments having a bend in common are disjoint.
- A rectangle in the neighborhood of each bend is added. This rectangle must intersect the two rectangles associated to the segments (Figure 4b). The three rectangles cannot intersect all mutually due to the previous point.

Finally, Figure 5 depicts the geometrical view produced from the rectilinear segmented embedding of Figure 3 by the previous polytime transformation. Notice
(a) Transformation of segments at a common endpoint.  (b) Transformation of segments having a bend in common.

Fig. 4: Atomic operations to transform the rectilinear segmented embedding view of a cubic planar graph into the geometrical view of a rectangle graph.

Fig. 5: The geometrical view obtained from the rectilinear segmented embedding of the graph of Figure 3.
Lemma 2. Let $G_p$ be a $n$-vertex $m$-edges cubic planar graph and an integer $k \leq n$. Let $G_r$ be the rectangle graph obtained by the previous transformation. The answer to CPVC with $(G_p,k)$ is yes iff the answer to RCP with $(G_r,k + 4 \times m)$ is yes.

Proof. This proof is illustrated in Figure 6.

Forward implication. Assume a $n$-vertex $m$-edges cubic planar graph $G_p$ has a vertex cover of cardinality $k$, denoted by the subset of $V_p$: $V' = \{v_1, \ldots, v_k\}$. We shall build a partition $\mathcal{P}$ of $G_r$, initially empty. To each edge $e = (v_i, v_j)$ of $E_p$ corresponds a 2-degree chain $c_e$ in $G_r$, with exactly nine vertices, from a vertex of the 3-clique $p_i$ (associated with $v_i$) to a vertex of the 3-clique $p_j$ (associated with $v_j$).

First, for every $v_i \in V'$, add the 3-clique $p_i$ in $\mathcal{P}$. Second, consider every edge $e$. By assumption the edge $e$ is covered by at least one vertex of $V'$. Precisely, the two following cases can be distinguished:

- $v_i \in V'$ or $v_j \in V'$ but not both. Then, one vertex of $c_e$ is already in $\mathcal{P}$. Add four 2-cliques in $\mathcal{P}$ by splitting the rest of the chain into pairs of connected vertices. For instance, in Figure 6, if we assume that $v_i$ is in $V'$, add $\{R_2, R_3\}$, $\{R_4, R_5\}$, $\{R_6, R_7\}$ and $\{R_8, R_9\}$.

- $v_i \in V'$ and $v_j \in V'$. Then, add the 4 following cliques into $\mathcal{P}$: $\{R_2, R_3\}$, $\{R_4, R_5\}$, $\{R_6, R_7\}$ and $\{R_8\}$.

This process covers each vertex of $G_r$ exactly once. Hence the set $\mathcal{P}$ forms a valid partition of $G_r$. Furthermore, The size of this partition is $k + 4 \times m$.

Backward implication.

Assume $G_r$ can be partition into $k + 4 \times m$ cliques and let $\mathcal{P}$ be such a partition. Consider an edge $e$ of $G_p$.

We shall distinguish three cases (note that the two first are not exclusive):
1. The clique that contains $R_1$ does not contain $R_2$. Then the clique that contains $R_2$ is either \{R_2\} or \{R_2, R_3\}. If it is \{R_2\}, we can remove $R_3$ from its clique and add it into \{R_2\}. This change can only decrease the total number of cliques in $\mathcal{P}$. Now that $R_2$ and $R_3$ belong to the same clique \{R_2, R_3\}, let us consider $R_4$. Repeating the same argument, we can make $R_4$ form a clique with $R_5$, and so on. Finally, we can rearrange $\mathcal{P}$ without increasing its cardinality so that it contains the sub-partition \{R_2, R_3\}, \{R_4, R_5\}, \{R_6, R_7\}, \{R_8, R_9\}.

2. The clique that contains $R_9$ does not contain $R_8$. By a symetrical reasoning, $\mathcal{P}$ can be modified so that it contains the following sub-partition: \{R_8, R_7\}, \{R_6, R_5\}, \{R_4, R_3\}, \{R_2, R_1\}.

3. The two previous conditions do not hold. Then, the clique containing $R_1$ (resp. $R_9$) is \{R_1, R_2\} (resp. \{R_8, R_9\}). Up to a replacement of $R_4$, we can assume that $R_3$ and $R_4$ form a clique and so on. Again, we can rearrange $\mathcal{P}$ so that it contains the following sub-partition: \{R_1, R_2\}, \{R_3, R_4\}, \{R_5\}, \{R_6, R_7\}, \{R_8, R_9\}.

Now, let us build the covering set $V'$ of $G_P$ as follows. For every edge $e = (v_i, v_j)$, if \{R_1, R_2\} is not in $\mathcal{P}$, add $v_i$ into $V'$. Similarly, if \{R_8, R_9\} is not in $\mathcal{P}$, add $v_j$ into $V'$. Otherwise, add any one of the two vertices ($v_i$ or $v_j$) into $V'$. The edge $e$ is then “covered” by $V'$. By looking to the three cases above, we see that when $e$ is done, we have added one vertex into $V'$ from a set of 5 cliques of $\mathcal{P}$. Hence, $V'$ is augmented with a number of vertices which is 4 less than the number of cliques associated to $e$.

When all the edges are done, all the edges are covered and the size of $V'$ is bounded by $k$. \hfill \Box

Lemma 1 and 2 leads to the following proposition:

**Proposition 1.** The rectangle clique partition problem (RCP) is NP-complete.

4 Conclusion

This paper provides a new result that extends our knowledge about complexity of problems related to intersection graphs.

However, the motivation of this work goes beyond the scope of graph theory. The problem under consideration appears in the context of constraint programming when generalizing the nvalue constraint \[2, 4\] to the multi-dimensional case.

Indeed, the size of the smallest clique partition for an intersection graph is also the minimum number of distinct vectors that can be obtained by taking one vector inside each object associated to the vertices.

A natural generalization of the nvalue constraint consists precisely in bounding this number in the case of variables varying into rectangles (or boxes of higher dimensions). Boxes are indeed the standard way of representing domains for continuous (i.e., not discrete) variables.

This note therefore sets the untractability of the generalized nvalue constraint in the continuous case. Such generalization is unpublished yet.
References