

Scheduling and SAT

Emmanuel Hebrard

The logo for LAAS-CNRS features the text "LAAS-CNRS" in a dark blue, sans-serif font. It is framed by two horizontal lines: a purple line above and a yellow line below.

LAAS-CNRS

Toulouse

Outline

- 1 Introduction
- 2 Scheduling and SAT Encoding
- 3 Scheduling and SAT Heuristics
- 4 Scheduling and SAT Hybrids
- 5 Conclusion

Outline

1 Introduction

- Preamble
- Scheduling Background
- SAT Background

2 Scheduling and SAT Encoding

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Scheduling with Boolean Satisfiability

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- Important theoretical results
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 - ▶ [Schaefer]’s dichotomy theorem
- Efficient algorithms (CDCL)
- Successful in Circuit design, Model checking, Planning, ...

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- All the approaches discussed here were developed in the last 5 years
- Recent progress in SAT algorithms opens new research opportunities

Scheduling Problems

Terminology

- Tasks (preemptive, non-preemptive)
- Resources (disjunctive, cumulative, reservoir,...)
- Objectives (makespan, tardiness, flow time,...)
- Side constraints (precedence, time windows, time lags,...)

Scheduling Problems

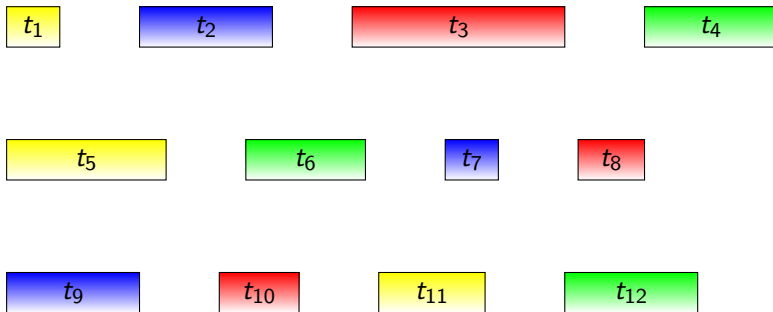
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Tip of the iceberg

- SAT-based methods have been applied to a very small subset scheduling problems.
 - ▶ Minimization of makespan for non-preemptive tasks and disjunctive resources

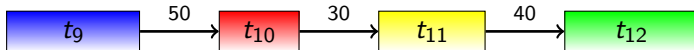
Jobshop Scheduling Problem



Problem description

- A set of non-preemptive tasks

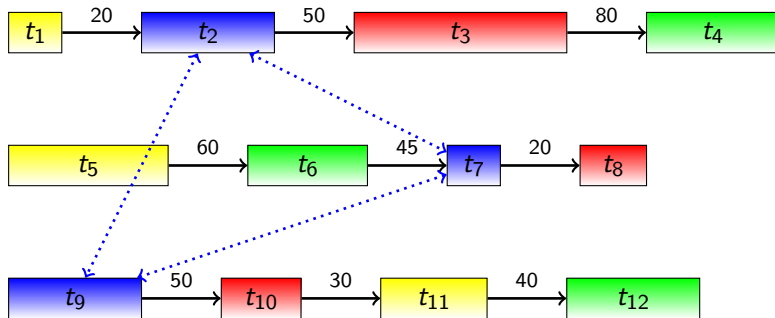
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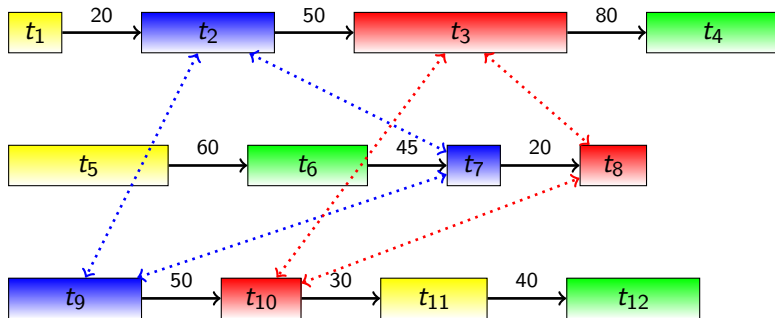
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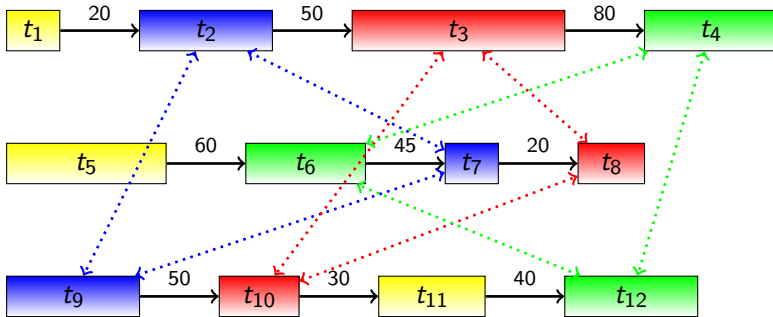
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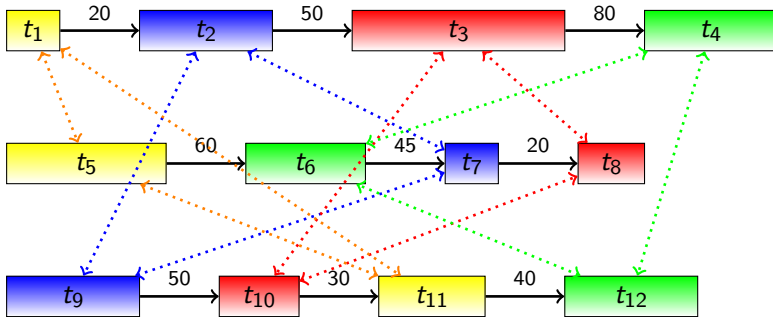
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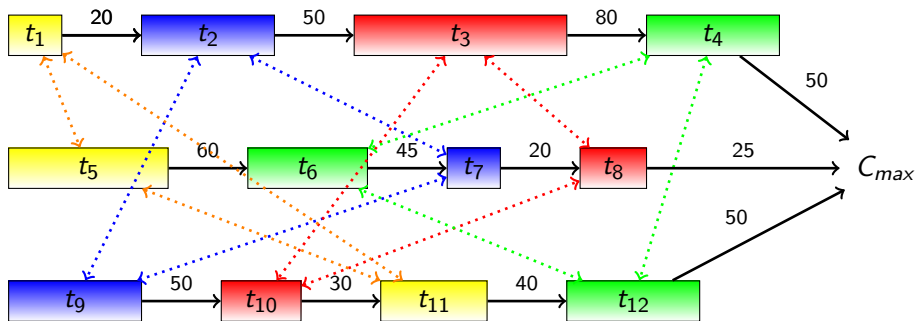
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- Objective: minimize the total duration (C_{max})

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 - ▶ Extract a learned clause
 - ▶ Backjump several levels and unit-propagate the learned clause
- Adaptive branching heuristics (weight conflicting literals)
- And also: restart, simplify the clause base, forget clauses, etc.

CDCL: Example

	f		

$$\bar{a} \vee \bar{f} \vee g$$

$$\bar{a} \vee \bar{b} \vee \bar{h}$$

$$a \vee c$$

$$a \vee \bar{i} \vee \bar{l}$$

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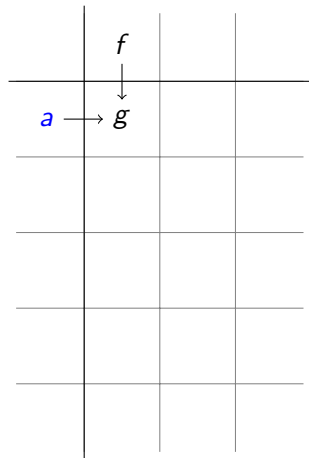
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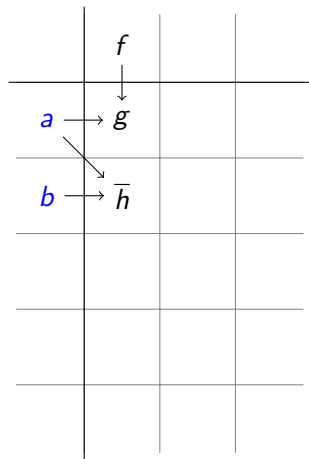
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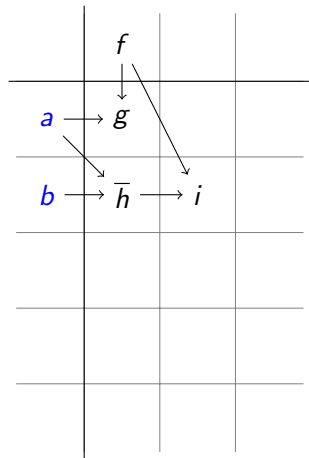
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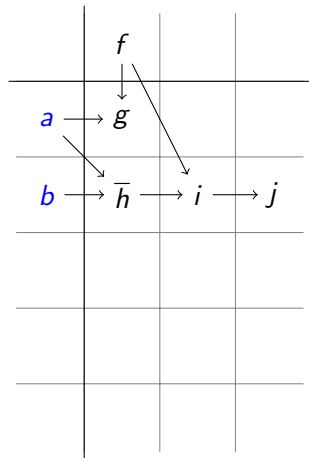
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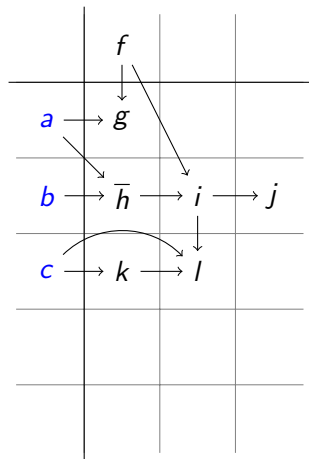
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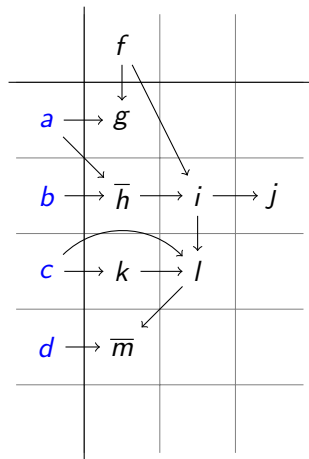
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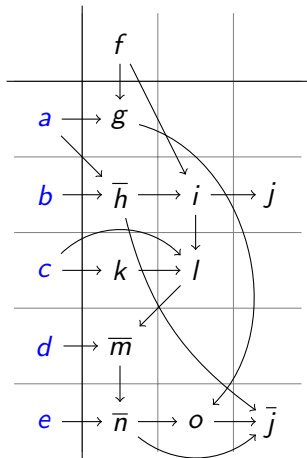
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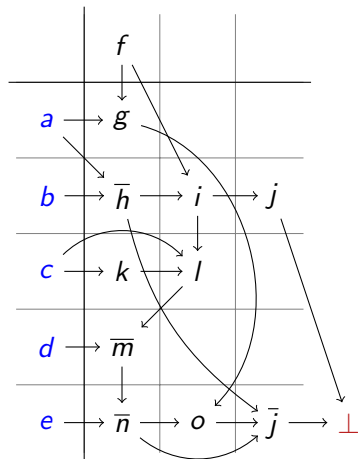
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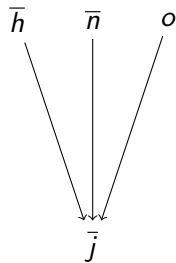
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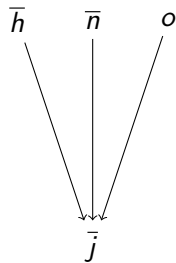
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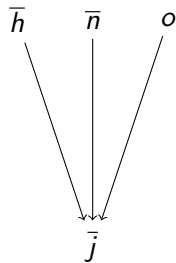
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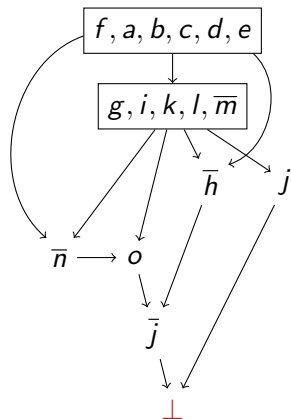
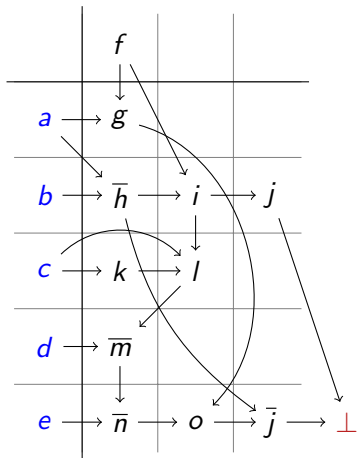
$$(h \vee \bar{o} \vee \bar{j} \vee n)$$

CDCL: Example

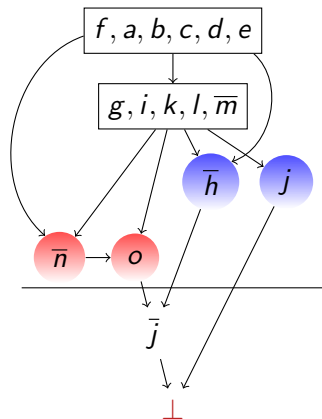
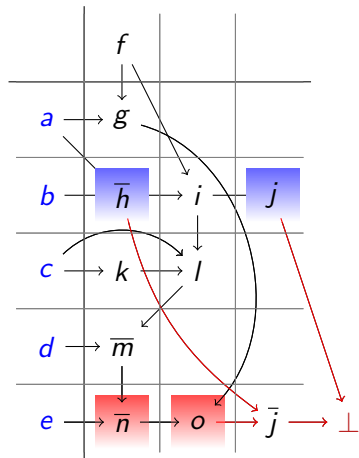


$$\begin{aligned} & (h \vee \bar{o} \vee \bar{j} \vee n) \\ & \equiv \\ & (\bar{h} \wedge o \wedge \bar{n}) \rightarrow \bar{j} \end{aligned}$$

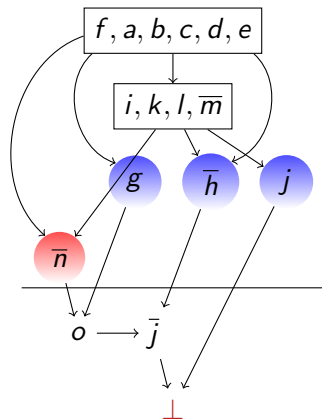
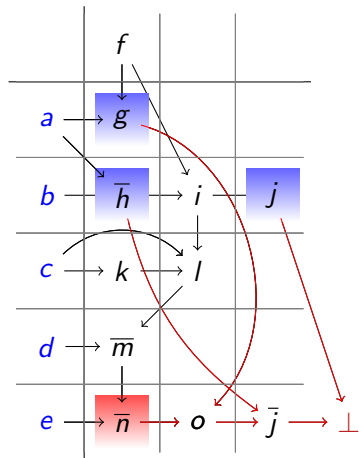
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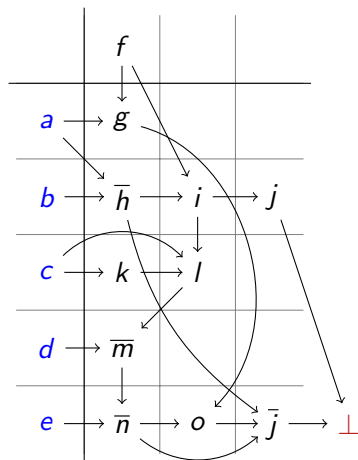
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$$\bar{a} \vee \bar{f} \vee g$$

$$\bar{a} \vee \bar{b} \vee \bar{h}$$

$$a \vee c$$

$$a \vee \bar{i} \vee \bar{l}$$

$$a \vee \bar{k} \vee \bar{j}$$

$$b \vee d$$

$$b \vee g \vee \bar{n}$$

$$b \vee \bar{f} \vee n \vee k$$

$$\bar{c} \vee k$$

$$\bar{c} \vee \bar{k} \vee \bar{i} \vee l$$

$$c \vee h \vee n \vee \bar{m}$$

$$c \vee l$$

$$d \vee \bar{k} \vee l$$

$$d \vee \bar{g} \vee l$$

$$\bar{g} \vee n \vee o$$

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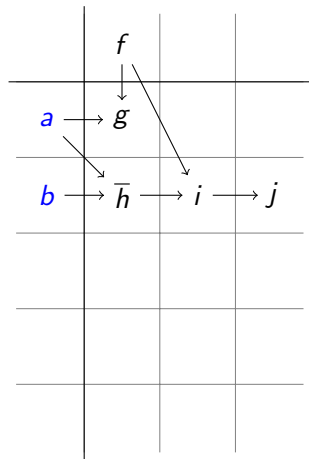
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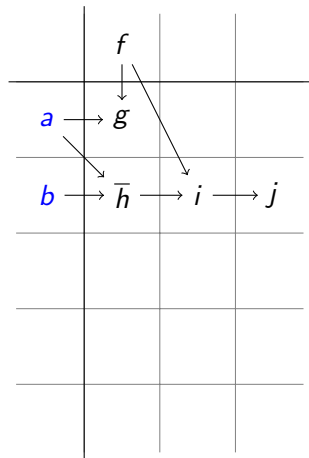
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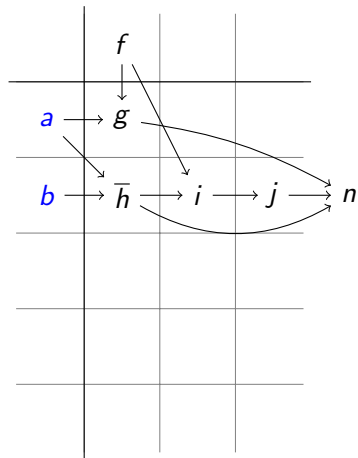
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Outline

1 Introduction

2 **Scheduling and SAT Encoding**

- Formulation into SAT
- Scheduling by encoding into SAT

3 Scheduling and SAT Heuristics

4 Scheduling and SAT Hybrids

5 Conclusion

CNF encoding

- The way we encode problems into SAT has a huge impact on efficiency
 - ▶ Encoding of Planning problems
 - ▶ Encoding of CSP (Direct, Log, AC-encoding)
 - ▶ Encoding of Pseudo-Boolean (Adder, Sorter)

Direct Encoding

Domain

- An atom i_v for each pair $(x_i, v \in D(x_i))$
 - ▶ $i_v \Leftrightarrow x_i = v$ (for instance: $2 \leftrightarrow 0100$)
- Must take at least a value: $i_1 \vee i_2 \vee \dots \vee i_n$
- Must take at most one value: $\bigwedge_{v \neq w \in D(x_i)} \overline{i_v} \vee \overline{i_w}$

Direct Encoding

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Complexity

- $O(n^2)$ space: $n(n-1)/2$ binary clauses and one n -ary clause.
- There are different ways to encode the constraints.

Constraints: Tuple Encoding

Example of constraint: $x_i < x_j$

x_i x_j	1	2	3	4
1	$\overline{i_1} \vee \overline{j_1}$	$\overline{i_2} \vee \overline{j_1}$	$\overline{i_3} \vee \overline{j_1}$	$\overline{i_4} \vee \overline{j_1}$
2		$\overline{i_2} \vee \overline{j_2}$	$\overline{i_3} \vee \overline{j_2}$	$\overline{i_4} \vee \overline{j_2}$
3			$\overline{i_3} \vee \overline{j_3}$	$\overline{i_4} \vee \overline{j_3}$
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Costly (in space) and weak (in propagation)

- $O(n^2)$ binary clauses.
- $\overline{i_4}(x_i \neq 4)$ and $\overline{j_1}(x_j \neq 1)$ are inconsistent, but not unit propagated.

Constraints: AC Encoding [Kasif 90]

Example of constraint: $x_i < x_j$

assignment	atom	support
$x_i = 1$	$\overline{i_1}$	$j_2 \vee j_3 \vee j_4$
$x_i = 2$	$\overline{i_2}$	$j_3 \vee j_4$
$x_i = 3$	$\overline{i_3}$	j_4
$x_i = 4$	$\overline{i_4}$	\perp

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$x_i = 3$	\bar{i}_3	j_4
$x_i = 4$	\bar{i}_4	\perp

Same space complexity, better propagation

- $O(n)$ n -ary clauses
- $\bar{i}_4(x_i \neq 4)$ and $\bar{j}_1(x_j \neq 1)$ are unit clauses.

Order Encoding [Crawford & Backer 94]

Domain

- An atom i_v for each pair $(x_i, v \in D(x_i))$
 - ▶ $i_v \Leftrightarrow x_i \leq v$ (for instance: $2 \leftrightarrow 0111$)
- Bound propagation:
 - ▶ If $x_i \leq v$ then $x_i \leq v + 1$
 - ▶ $\bigwedge_{v \in D(x_i)} \overline{i_v} \vee i_{v+1}$

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Complexity

- $O(n)$ space ($n - 1$ binary clauses)

Constraints: BC Encoding

Example of constraint: $x_i < x_j$

relation	clause
$x_i > 0 \Rightarrow x_j > 1$	$\perp \vee \overline{j_1}$
$x_i > 1 \Rightarrow x_j > 2$	$i_1 \vee \overline{j_2}$
$x_i > 2 \Rightarrow x_j > 3$	$i_2 \vee \overline{j_3}$
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Better complexity and same propagation on some linear constraints

- $O(n)$ space (n binary clauses)
- $i_3(x_i \leq 3)$ and $\overline{j_1}(x_j > 1)$ are unit clauses.

Domain

- An atom i_k for each value in $[1, \dots, \lfloor \log_2 ub \rfloor]$ (assuming $D(x) = [0, \dots, ub]$)
 - ▶ $\sum_{k=1}^{ub} 2^k * i_k = v \Leftrightarrow x_i = v$ (for instance: $2 \leftrightarrow 01$)
- For interval domains, no need for extra clauses

Log Encoding [Walsh 00]

Domain

- An atom i_k for each value in $[1, \dots, \lfloor \log_2 ub \rfloor]$ (assuming $D(x) = [0, \dots, ub]$)
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Complexity

- $O(\log_2 n)$ space

Propagation

- Encoding constraints is trickier, and less powerful

Other Encodings

Many more!

- Mix of direct and order encoding [lazy-FD, Numberjack]
- Mix of AC and log encoding [Gavanelli 2007]
- Mix of order and log encoding [Sugar, Tamura et al. 2006]

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- Mix of direct and order encoding [lazy-FD, Numberjack]
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- Mix of order and log encoding [Sugar, Tamura et al. 2006]
 - ▶ Log encoding in a base B and order encoding inside a digit
 - ▶ Excellent results on scheduling benchmarks! (with CDCL solvers)

Order Encoding, Now and Then

Progress of SAT solvers

- From a few hundreds variables in the 90's to millions now

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- The hardest instance requires a few 100s conflicts at the most

Closing the Open Shop

Instances

- [Gueret & Prins]: hard for local search, extremely easy for SAT/CP
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results

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 - ▶ The two hardest instances were decomposed into 120 subproblems, and required up to 13h to solve
- First approach to close the open shop!

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- [Tamura et al.]'s encoding is better than order encoding
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- It is now possible to efficiently solve some scheduling problem simply by formulating it as a CNF formula

Outline

1 Introduction

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3 Scheduling and SAT Heuristics

- A SAT-like Approach
- Comparison with the State of the Art

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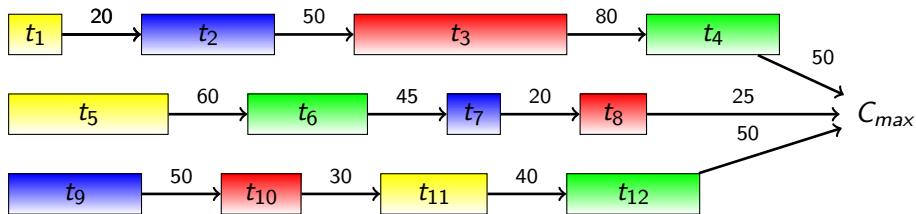
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- Are adaptive heuristics all that we need to solve disjunctive scheduling problems?

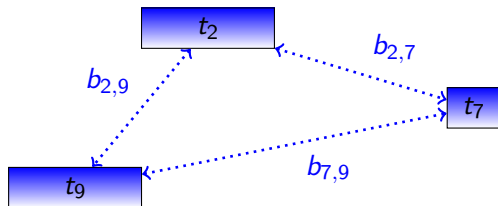
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- A Boolean Variable standing for the relative order of each pair of conflicting tasks (disjunct):
 - ▶ Binary Disjunctive constraints: $b_{ij} = \begin{cases} 0 \Leftrightarrow t_i + p_i \leq t_j \\ 1 \Leftrightarrow t_j + p_j \leq t_i \end{cases}$

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Restarts

- Geometric [Walsh 99], nogoods on restarts [Lecoutre et al. 07]
- Almost no problem specific method

CP or SAT?

- Many similarities with SAT:
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- Faster propagation, but no clause learning
- Restarts + weighted degree “simulates” CDCL behavior?

Experiment on Jobshop and Variants

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 - ▶ Simple decomposition to express the new objective

Experimental Protocol

- This simple model was run on several standard benchmarks
 - ▶ 1 hour cutoff
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- Best known results on each benchmark (LS, CP, MIP)
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- ▶ Negative: how much worse than M (when it is)

Experimental Protocol

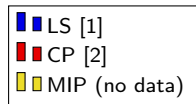
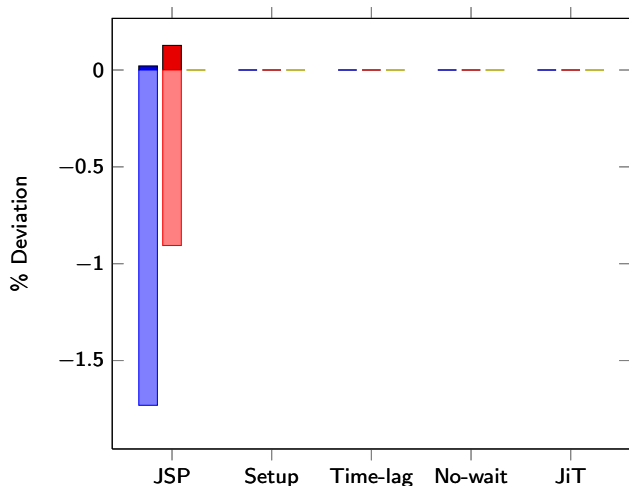
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▶

$$100 \times \sum_{\text{instance } x} \frac{M \text{ objective}(x) - SAT \text{ objective}(x)}{\#instances \times best \text{ objective}(x)}$$

- ▶ Negative: how much worse than M (when it is)
- ▶ Positive: how much better than M (when it is)

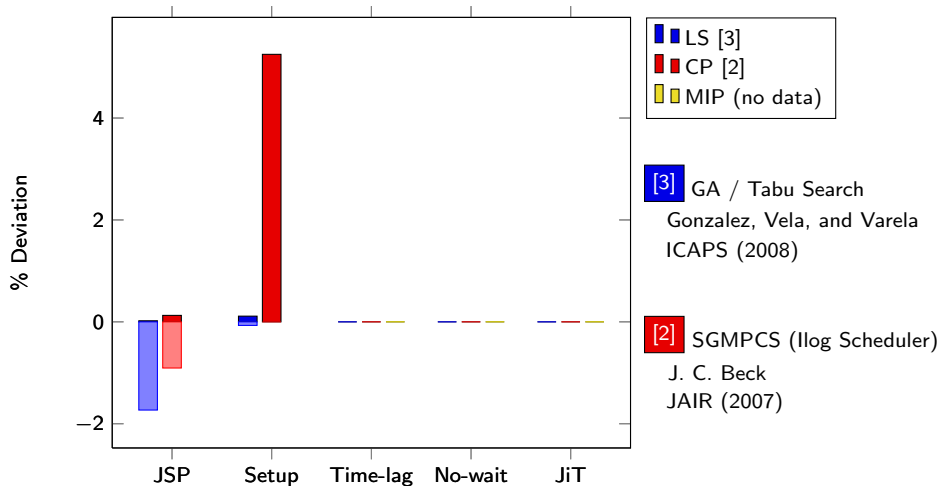
Jobshop - C_{max} - Taillard



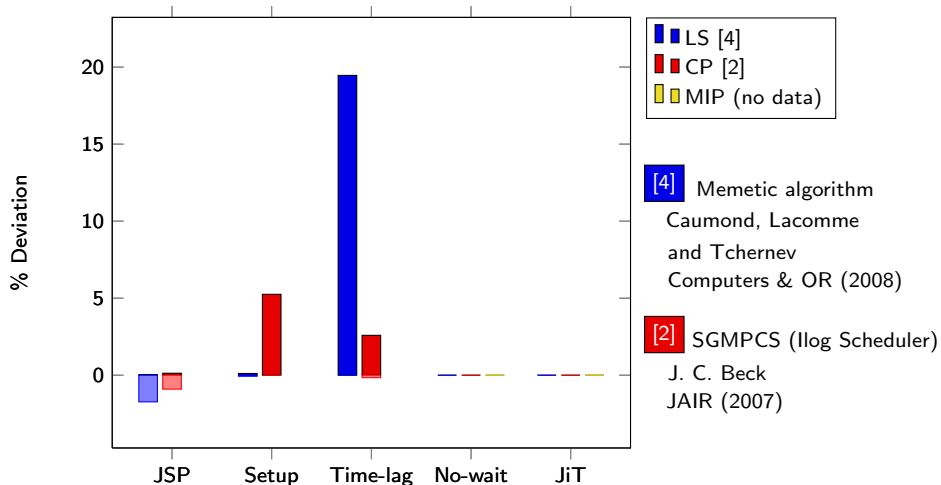
[1] i-TSAB (Tabu Search)
E. Nowicki and C. Smutnicki
J. of Scheduling (2005)

[2] SGMPCS (Ilog Scheduler)
J. C. Beck
JAIR (2007)

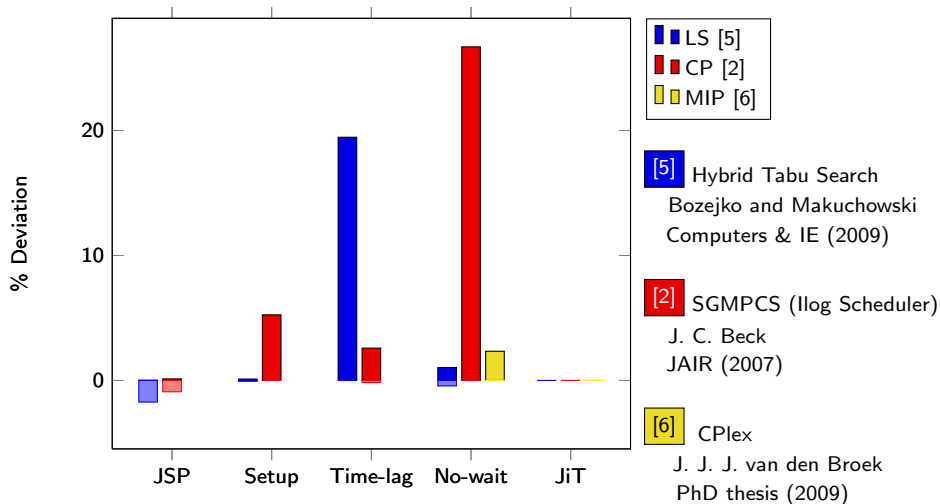
Jobshop with setup times - C_{max} - Brucker & Thiele



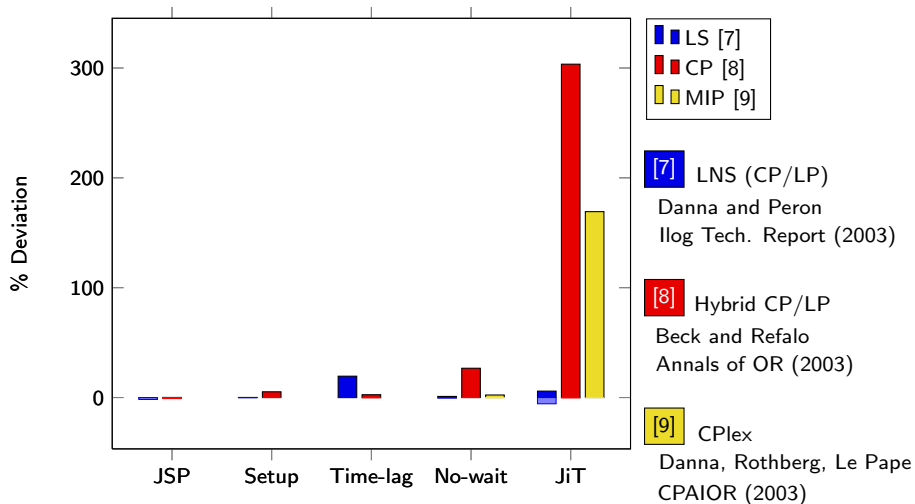
Jobshop with time lags - C_{max} - Lawrence (modified)



“No-wait” Jobshop - C_{max} - Lawrence



Jobshop - earliness/tardiness - Beck & Refalo; Morton & Pentico



SAT Strategies

- Often comparable or better than the state of the art

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 - ★ Even this “pseudo” learning helps!

Outline

- 1 Introduction
- 2 Scheduling and SAT Encoding
- 3 Scheduling and SAT Heuristics
- 4 Scheduling and SAT Hybrids**
 - Lazy clause generation
 - Satisfiability Modulo Theories
- 5 Conclusion

SAT Hybrids

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- Lazy Clause Generation
- SAT Modulo Theories

Lazy Clause Generation [Ohrimenko, Stuckey & Codish 07] - [Feydy & Stuckey 09]

Architecture

- Channel a CP and SAT representations
 - ▶ Search and propagation in CP
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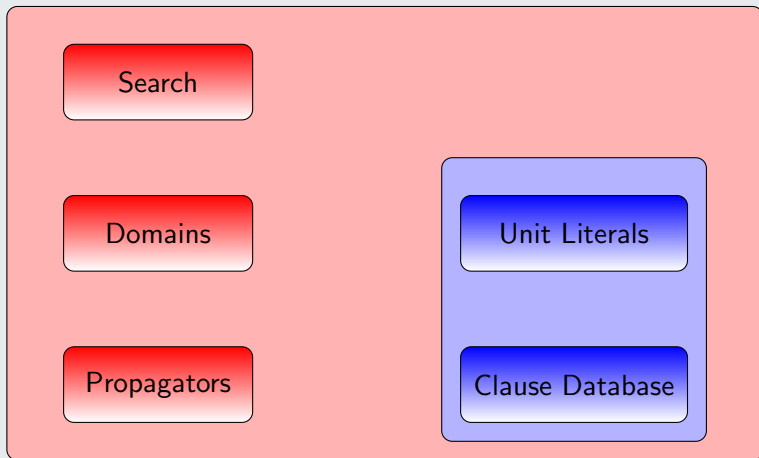
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Architecture

- Channel a CP and SAT representations
 - ▶ Search and propagation in CP
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 - ▶ The SAT formulation is generated lazily (learned during search)

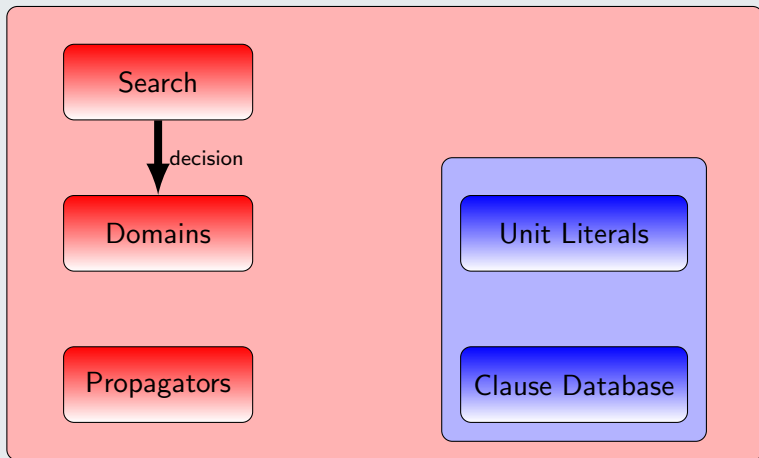
Lazy Clause Generation

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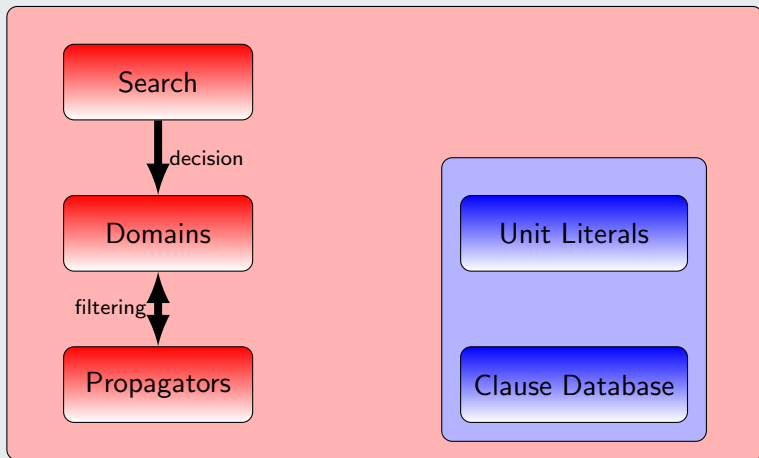
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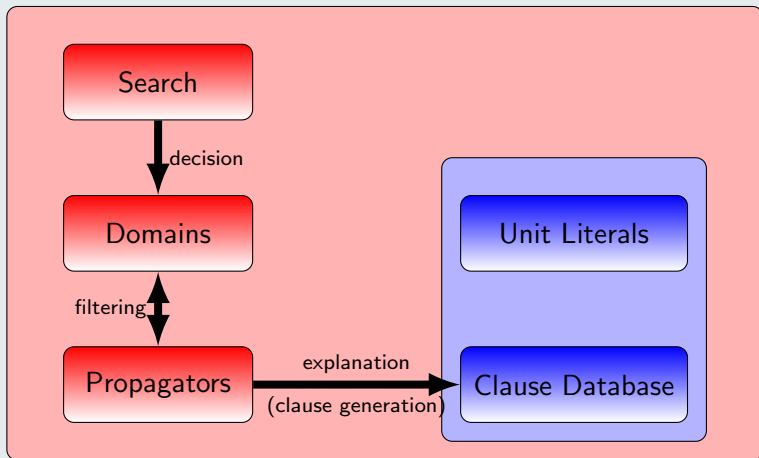
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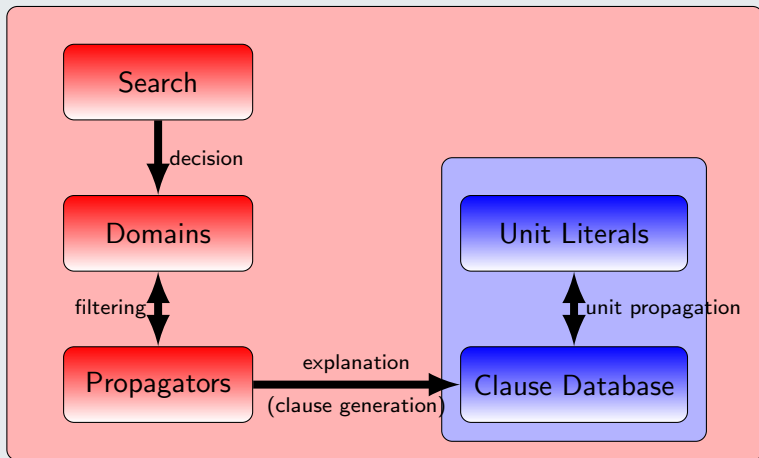
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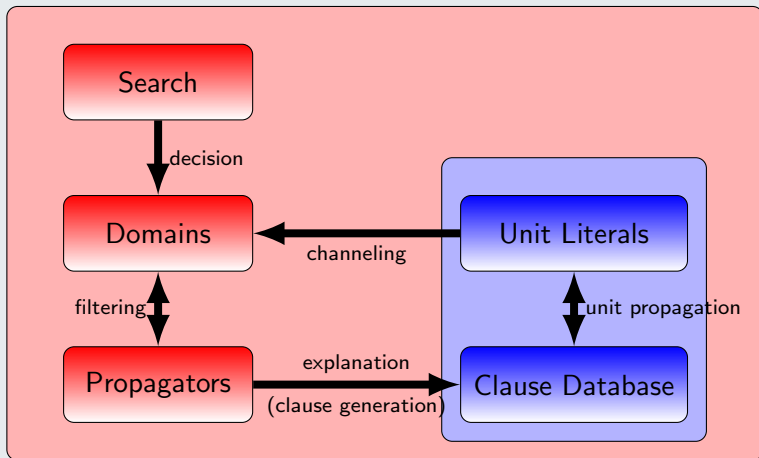
Lazy Clause Generation

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Lazy Clause Generation

Architecture



Lazy-FD: Example

$$x_i < x_j$$

- Initial representation

	CP view	SAT view
$D(x_i)$	$\{1, \dots, 4\}$	$\overline{i_1} \vee i_2, \overline{i_2} \vee i_3$
$D(x_j)$	$\{2, \dots, 5\}$	$\overline{j_2} \vee j_3, \overline{j_3} \vee j_4$
constraint	$x_i < x_j$	
constraint	$C(x_i, x_k, \dots)$	

Lazy-FD: Example

$$x_i < x_j$$

- Some constraint reduces the domain of x_i to $\{2, \dots, 5\}$

	CP view	SAT view
$D(x_i)$	$\{2, \dots, 4\}$	$\bar{i}_1 \vee i_2, \bar{i}_2 \vee i_3$
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Lazy-FD: Example

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- An explanation clause $T \vee \bar{i}_1$ is produced, and the unit literal \bar{i}_1 is propagated

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Lazy-FD: Example

$$x_i < x_j$$

- The propagator for $x_i < x_j$ is triggered and reduces the domain of x_j

	CP view	SAT view
$D(x_i)$	$\{2, \dots, 4\}$	$\bar{i}_1 \vee i_2, \bar{i}_2 \vee i_3$
$D(x_j)$	$\{3, \dots, 5\}$	$\bar{j}_2 \vee j_3, \bar{j}_3 \vee j_4$
constraint	$x_i < x_j$	
constraint	$C(x_i, x_k, \dots)$	$T \vee \bar{i}_1$

Lazy-FD: Example

$$x_i < x_j$$

- An explanation clause is also produced

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constraint	$x_i < x_j$	$i_i \vee \overline{j_2}$
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Results on Resource Constrained Project Scheduling Problem (RCPSP) [Schutt, Feydy, Stuckey & Wallace 09]

Resource Constrained Project Scheduling Problem (RCPSP)

- Cumulative resources, each task has a demand r_k for the resource k

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- Favorable comparison with state of the art approaches
 - ▶ MCS (implemented on top of Ilog-Scheduler [Laborie 05])
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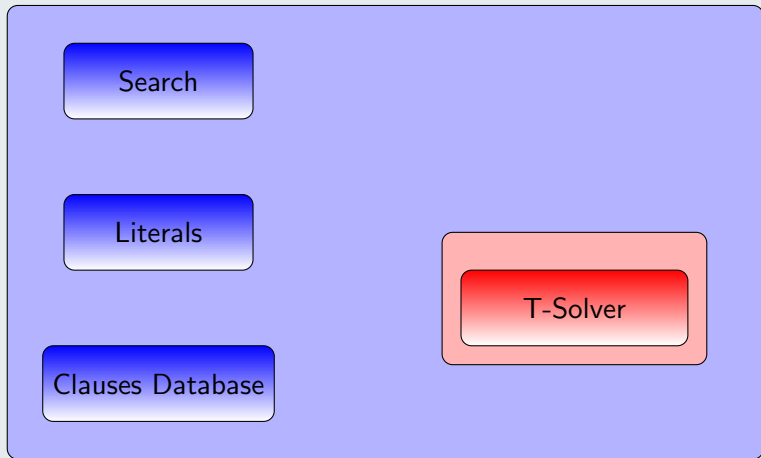
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- 54 open instances closed!

SAT Modulo Theories (SMT)

- Framework to hybrid dedicated solvers (Theories, or T-Solvers) with CDCL solvers
 - ▶ T-Solver view: a set of propositions each represented by a literal in F
 - ▶ CDCL-Solver view: a CNF formula F partially representing the problem
- CDCL-Solver makes decisions and analyzes the conflicts
- T-Solver detects conflicts and/or propagates and generates explanation clauses

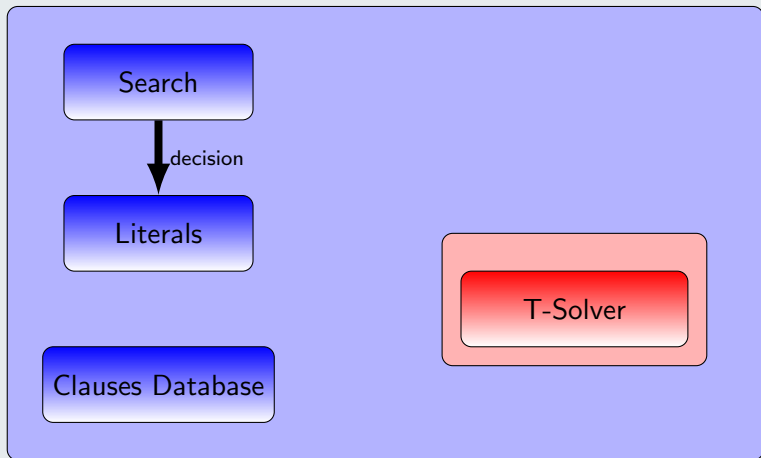
SMT Solver

Architecture



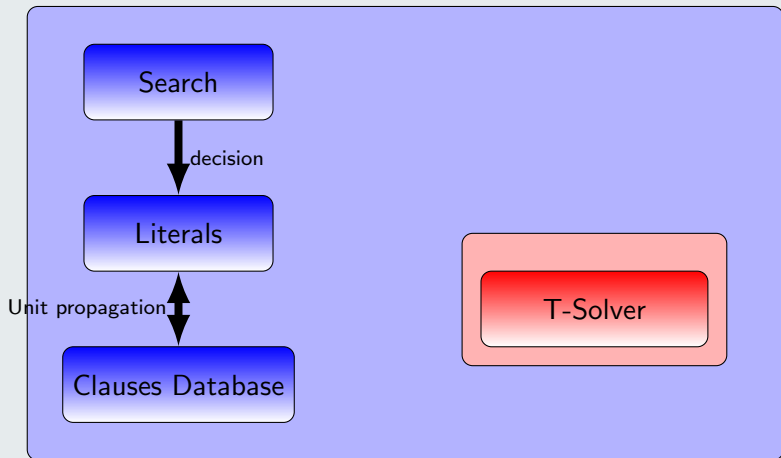
SMT Solver

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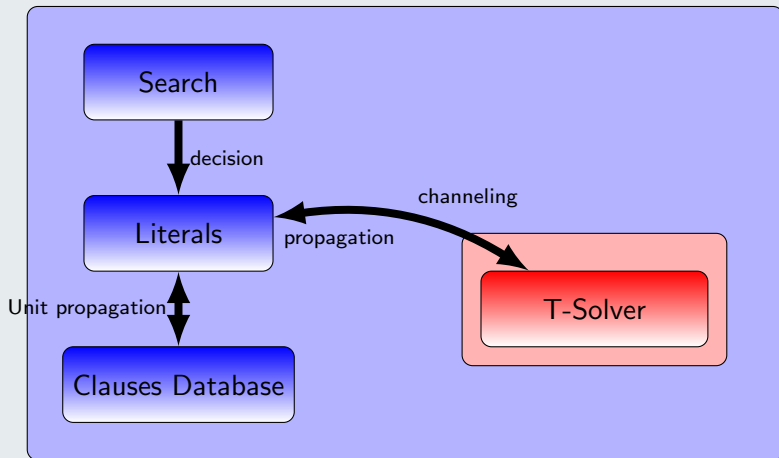
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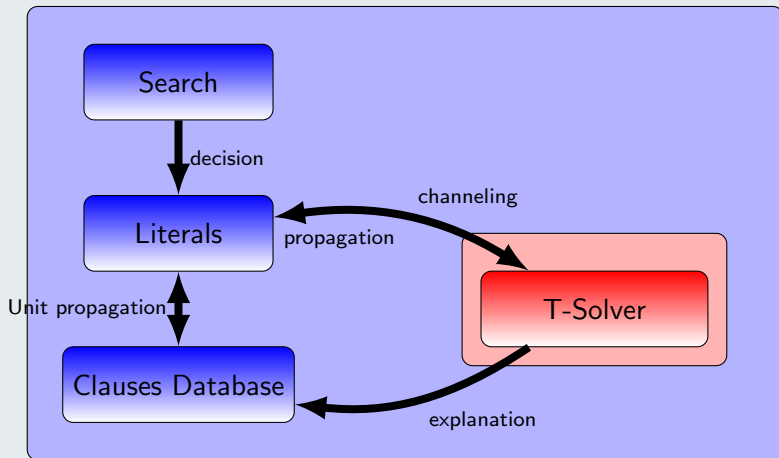
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Several Theories

T-Solvers

- Linear Real Arithmetic,
- Arrays,
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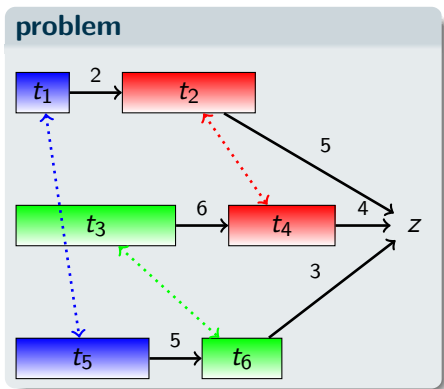
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SMT for scheduling

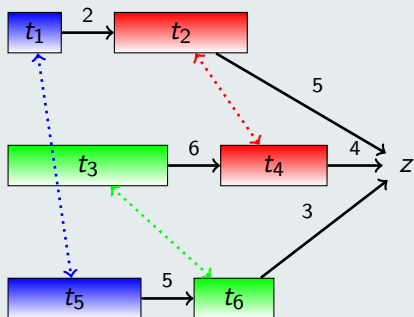
- Satisfiability Modulo Difference Logic.

Example: Jobshop Scheduling



Example: Jobshop Scheduling

problem



T-Solver view

$$s_1 - s_2 \leq -2$$

$$z - a \leq 15$$

$$s_3 - s_4 \leq -4$$

$$s_5 - s_6 \leq -5$$

$$s_2 - z \leq -5$$

$$l_{1 \rightarrow 5} \Leftrightarrow s_1 - s_5 \leq -2$$

$$s_4 - z \leq -4$$

$$l_{5 \rightarrow 1} \Leftrightarrow s_5 - s_1 \leq -5$$

$$s_6 - z \leq -3$$

$$l_{2 \rightarrow 4} \Leftrightarrow s_2 - s_4 \leq -5$$

$$a - s_1 \leq 0$$

$$l_{4 \rightarrow 2} \Leftrightarrow s_4 - s_2 \leq -4$$

$$a - s_3 \leq 0$$

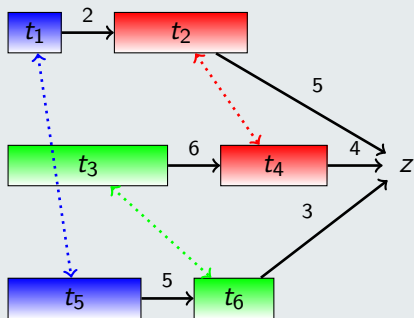
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CDCL-Solver view

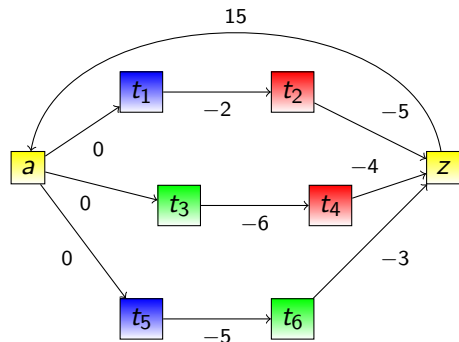
$$l_{1 \prec 5} \vee l_{5 \prec 1}$$

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Example: Jobshop Scheduling

- Reasoning: detection of negative cycles ([Bellman-Ford])



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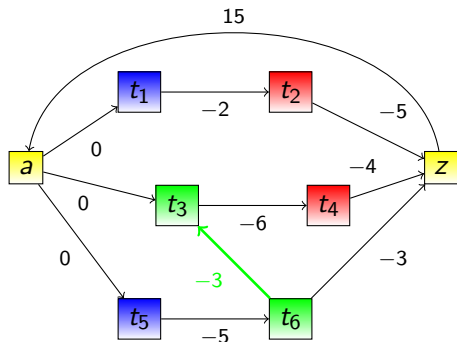
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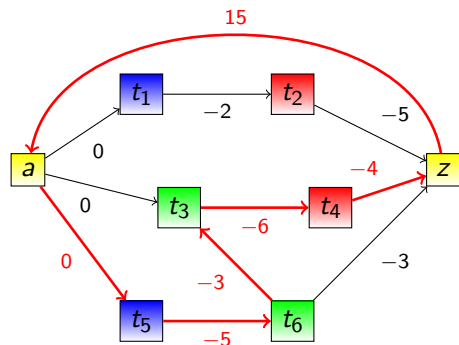
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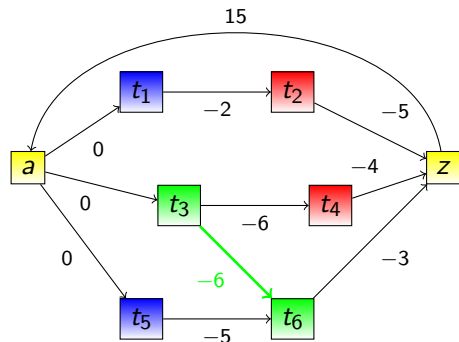
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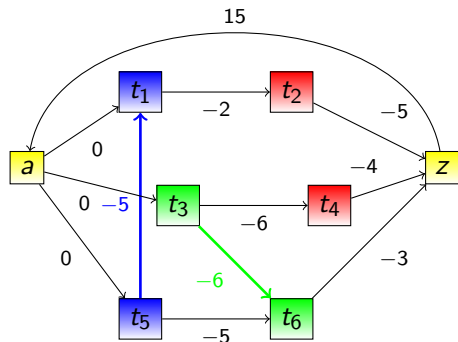
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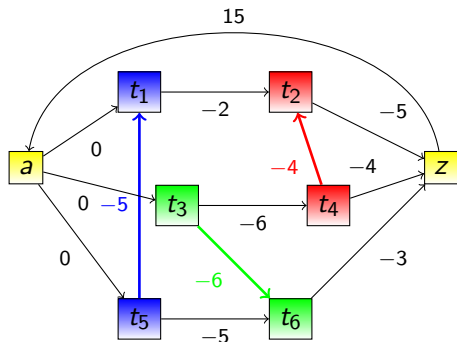
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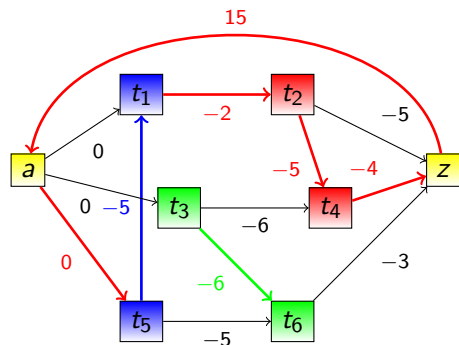
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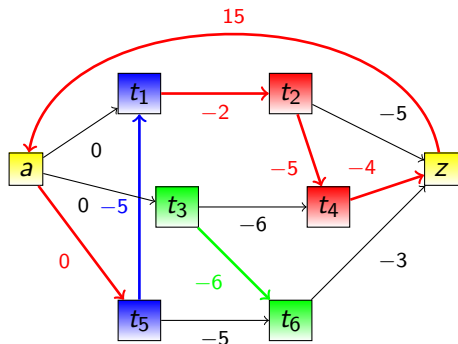
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Learned clause

- $\overline{l_{5 \prec 1}} \vee \overline{l_{2 \prec 4}}$

Results on Resource Constrained Project Scheduling Problem (RCPSP) [Ansótegui et al. 11]

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Two formulations

- Time encoding
- Task encoding

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- Nogood learning [Schiex & Verfaillie 93] and explanation for global constraints [Rochart & Jussien 03], disjunctive resource [Vilím 05]?
 - ▶ Somehow it does not have the same impact as in SAT
- Hybridization (learning + dedicated reasoning) is the way to go
 - ▶ SAT Modulo Theories?
 - ▶ CDCL with global constraints and integer domains?
 - ▶ Explanation algorithms for global constraints?